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The Sources of Eratosthenes' Measurement of the Earth

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SUMMARY

Ancient Greek geographers expressed the north–south coordinate of a point in at least two different ways before the use of latitude became standard. Coordinates expressed in these ways were naturally rounded to convenient values. Later, when latitude was adopted, its values were often calculated from these rounded values instead of being measured afresh. This process may be the origin of the angular data that Eratosthenes used to estimate the size of the Earth.

ERATOSTHENES' MEASUREMENT

Ancient Greek scientists made several estimates of the circumference of the Earth, but the most famous one was made by Eratosthenes *circa* 300 BC. The writing of Eratosthenes is now lost except for fragments that were quoted (or more likely paraphrased) by later but still ancient writers. While the ancient writers sometimes disagree about what he actually wrote or did, they all luckily agree about the general method by which he estimated the circumference of the Earth. Dreyer (1905, pp. 174ff) and Fischer (1975) discuss the ancient writings connected with this estimate in considerable detail, and they give the necessary references to the ancient literature.

Eratosthenes believed that the Sun was directly overhead at noon on the longest day of the year at Syene (the modern Aswan) in Egypt. On this day in Alexandria, however, he believed that the zenith distance of the Sun was 1/50 of a circle. Since the distance between the two places, which he thought were on the same meridian, was 5000 stades, the circumference was therefore 250 000 stades.

Stade is the form used in modern English to denote the ancient unit of length that is called *stadium* in Latin and *stadion* in Greek. The literature in this century has seen a violent controversy about the length of the stade and indeed about whether there may have been many different *stadia*. Dicks (1960, p. 43) says that Lehmann-Haupt, in a paper that I have not consulted, gives evidence for at least six different types of stade, with lengths ranging from about 150 metres to about 210 metres. I reserve judgment about the validity of this evidence.

The years have also seen a violent controversy about the accuracy of Eratosthenes' estimate. The accuracy is, of course, intimately connected with the length of the stade. I have seen an accuracy of 0·5 per cent claimed in some recent sources, even though such an accuracy was far beyond the capability of measurement in the time of Eratosthenes.

In this paper, I do not intend to enter into this controversy. Instead, I shall advance an hypothesis about the origins of the various angles or arcs that are involved, either explicitly or implicitly, in Eratosthenes' estimate. If my hypothesis is correct, the controversy about the length of the stade is largely meaningless in this context, although it may be significant in other contexts.

THE ANGLES OR ARCS INVOLVED

Four different statements about angles or arcs are involved in Eratosthenes' estimate of the circumference of the Earth. They are:

(1) Aswan was located on the northern tropic (the Tropic of Cancer). In fact the latitude of the tropic in Eratosthenes' time was about 23° 43′ and the latitude of Aswan is 24° 5′. The error in saying that Aswan was on the tropic is about 22 arcmin, which is greater than the radius of the Sun.

(2) If Aswan was on the tropic, its latitude was the same as the obliquity of the ecliptic. According to Ptolemy (*circa* 142, Chapter I. 12), Eratosthenes took the obliquity to be 23° 51′ 20″. Therefore Eratosthenes took this value to be also the latitude of Aswan.

(3) By implication, Eratosthenes measured the latitude of Alexandria, and he found it to be 31° 3′ 20″. The correct value is about 31° 13′, so the error is about 10 arcmin. Note that the latitudes which Eratosthenes used for both Alexandria and Aswan are too small by large amounts. Note also that Ptolemy (*circa* 142, Chapter V. 12) implies that he measured the latitude of Alexandria and found it to be 30° 58′.

(4) Eratosthenes took the difference between the latitudes of Aswan and Alexandria and found it to be 1/50 of a circle, or 7° 12′.

The first three statements deal with the latitudes of the Tropic of Cancer, of Aswan, and of Alexandria. All are in error by an amount that is comparable to the radius of the solar disc, and the error in the first statement in fact exceeds this radius. We must look at methods of finding latitude that were available in ancient times in order to see the significance of this fact.

ANCIENT MEASUREMENTS OF LATITUDE

Ptolemy (*circa* 142, Chapter I. 12) describes a simple way of measuring latitude and of finding the obliquity of the ecliptic at the same time. One makes a graduated arc of a circle, such as a quadrant, and anchors it so that it lies in the plane of the meridian. One then puts a pin at its centre, so that the shadow of the pin will fall on the circle at noon. In this way one can measure the zenith distance of the Sun when it is in the meridian on any day that the weather permits.

To find the latitude and obliquity, one measures the zenith distance at the two solstices. The obliquity ϵ is half the difference of these values and the latitude β is half their sum. The accuracy of this method is rather high, particularly at a location as far south as Alexandria. There are four principal sources of error.

In order to measure the zenith distance of the Sun, one fits a marker of some sort to the graduated circle and adjusts it so that it is symmetrical with respect to the shadow. Since this is a symmetry operation, it can be done with high accuracy. I have done some testing on myself and some friends, and I believe that the standard deviation in locating the marker is about 1 part in 50 of the diameter. This is about 40 arcsec in the case of the Sun.

The second source is instrumental. The circle must be graduated and aligned accurately. The accuracy of doing this depends upon the size of the circle and the care of the maker, among other things, but the operations involved are still essentially those of symmetry, and it is not likely that they exceed 1 arcmin.

The third is refraction. The amount of refraction at Alexandria is 7 arcsec at the summer solstice and 1′ 25″ at the winter solstice. The error in the latitude is the average of these, namely 46 arcsec.

Finally we have the error in reading the position of the marker on the circle. On the basis of my own experience, I believe that the standard deviation of such a reading is about 1 part in 16 or less. If the circle is graduated only to the nearest degree, the resulting error is slightly less than 4 arcmin.

Overall, I think we can safely say that the standard deviation of a single reading of the Sun's zenith distance was about 4 arcmin. Since the latitude is the average of two readings, its standard deviation was about 3 arcmin. In contrast, Eratosthenes made an error of 22 arcmin in locating Aswan with respect to the tropic and he made an error of 10 arcmin in the latitude of Alexandria, while Ptolemy made an error of 15 arcmin in the same latitude. It does not seem possible that these errors came from using a graduated arc to read the zenith distance of the Sun.

An older way to find the zenith distance of the Sun was to measure the length of the shadow cast at noon by a vertical rod called a gnomon. The length of the shadow divided by the length of the gnomon equals what we call the tangent of the zenith distance. Here we have an unsymmetrical situation and the errors are much larger. The largest error probably comes from the fact that the Sun is not a point source and the shadow does not have a sharply defined edge. In using a gnomon, it is easy to make an error equal to the radius of the Sun, but a much larger error is unlikely.

According to ancient tradition (Fischer, 1975), Eratosthenes found the zenith distance of the Sun by using a vertical rod placed in a hemispherical bowl. This device gives the zenith distance directly instead of giving its tangent, but it is still subject to the error that arises from the finite size of the Sun. It seems doubtful to me that this device is appreciably more accurate than the conventional gnomon whose shadow length is measured.

The size of the error in relating Aswan to the tropic is 22 arcmin, which exceeds the radius of the Sun by a considerable amount. No known method of observation is likely to yield such an error. On the other hand, if Eratosthenes used 31° 3′ 20″ for the latitude of Alexandria (I shall give a reason below for suspecting that he used 30° 58′), he made an error of only about 10 arcmin and Ptolemy, who did use 30° 58′, made an error of about 15 arcmin. The sizes of both errors are consistent with the use of a gnomon, but this does not make it likely that either in fact did so, for a well-known reason that I shall discuss briefly in the next section. Further, Ptolemy claims rather explicitly that he used a graduated circle to make the measurement.

ACCIDENTAL AGREEMENT AND STATISTICAL SIGNIFICANCE

Before going on to suggest an origin for Eratosthenes' angular data, we must look briefly at a rather elementary point. Suppose that we make a measurement of some physical quantity, such as the latitude of Alexandria, and suppose that we find the value x. Suppose further that we believe that the standard deviation of the measurement equals σ. Then we often express the result by saying that the measured value is $x \pm \sigma$.

It often happens that we do not have any theoretical way to predict the value of the quantity when we make the measurement but that we discover a theory some time later. Let us say that the theory predicts the value X, which agrees reasonably well with x. We then ask: Is this agreement significant, or could it have come about simply by accident in the measuring process?

The significance clearly depends upon the ratio $|x - X|/\sigma$. If this ratio is large, it is not likely that the theoretical value is correct. If the ratio is small, we say that the agreement is statistically significant, because it is unlikely that such a close agreement would happen by chance. There are standard tables for calculating the probability that a particular value of the ratio would happen by chance.

In dealing with ancient astronomical data, we often have occasion to reverse the process. In the case of the latitude of Alexandria, for example, we know the correct value, call it X, rather accurately from modern results. We also have an ancient measurement, x say, and we have an ancient but perhaps mistaken description of how the measurement was made, from which we can estimate the standard deviation σ of the measuring process. We then use the ratio $|x - X|/\sigma$ to assess the probability that the process described was the one actually used. The process was probably the one used if the ratio is small, and it was probably not the one used if the ratio is large.

I shall apply this idea to Eratosthenes' data later in this paper, but it is useful at this point to look briefly at the latitude of Alexandria that Ptolemy claims to have found by using a graduated circle. As we saw above, σ for the process that he describes is about 3 arcmin, but the error in his result is about 15 arcmin, about 5σ. The probability that an error could be this large is less than 10^{-6}. Thus there is little chance that Ptolemy found the latitude by the method he describes; he almost surely found it in some other way. I shall suggest a possible origin for his value in the next section.

THE ORIGIN OF ERATOSTHENES' ANGULAR DATA

To find what I suggest as the origins of the angular data used by Eratosthenes, we must turn to some quantities that the ancient Greeks used to specify the north–south coordinate of a point. Apparently they did not use the latitude itself in the early development of Greek astronomy and geography. The first quantity they used was probably the shadow length of a gnomon that we have already discussed. The ratio of the shadow length to the length of the gnomon, if the measurement is made at an equinox, is just the tangent of the latitude.

Another quantity used was the length of the longest day of the year. In Chapter II. 6 of his work on astronomy, Ptolemy presents a table in which the independent variable is the length of the longest day, for day lengths ranging from 12 hours to 24 hours. The tabular interval varies from ¼ hour for the smallest values to 1 hour for the largest ones. For each length of the longest day, Ptolemy lists the latitude and the shadow lengths (for a gnomon of length 60) at noon at the equinoxes and the solstices.

The key to the situation, as Rawlins (1980) shows, is that ancient geographical tables were intended primarily for use by astrologers, whose calculations involved the position of a person's birthplace, among other things. The astrologers needed tables whose independent variable was one of the quantities used to specify the north–south coordinate. Since they did not need the position to high accuracy, they rounded the length of the longest day, say, to one of the values given in whatever table they happened to be using. Later, when latitude itself was adopted as a coordinate, geographers (or astrologers) calculated it from such a rounded value, without realizing that it was a rounded value and not a directly measured one. This explains why the typical error in a latitude taken from an ancient geography is a degree.

This process suggests what may be the origins of the angular data that Eratosthenes used. There are four points involved in my suggestion.

(1) Placing Aswan on the tropic. This is probably the easiest point to understand, even though we cannot provide a unique explanation of it. One possibility is that travellers who had been to Aswan told their friends when they got back to Alexandria that there were no noontime shadows at Aswan on the longest day, and a perspiring traveller looking down at his own noontime shadow there could indeed be excused for saying that he had none. There are numerous ways by which people could have been led to believe that Aswan was on the tropic; all involve taking a vague observation and transforming it into a precise statement.

(2) The length of the longest day at Aswan. The correct value was about $13^h\,31^m$*, which would naturally be rounded to 13½ hours; no one at the time would have considered this to be a serious error. According to Ptolemy's table (Ptolemy, *circa* 142, Chapter II. 6), 23° 51′ 20″ is the value of the latitude that corresponds to a longest day of 13½ hours; I have not verified the calculation. Thus taking the longest day as 13½ hours led to taking 23° 51′ 20″ as being simultaneously the latitude of Aswan and the obliquity of the ecliptic†.

*I should point out that the length of the day was almost surely not a measured quantity. It was probably calculated from a measurement of the north–south coordinate, which might have been made either by a gnomon or by a graduated circle. Refraction and the apparent diameter of the Sun would have been ignored in the calculation.

†The matter is complicated by the fact that the longest day at any place depends upon the obliquity as well as upon the latitude of the place. At Aswan we have the unique situation where the latitude must equal the obliquity and where the length of the longest day is to come out as 13½ hr.

(3) The length of the equinoctial noontime shadow at Alexandria. The length of this shadow is approximately 36·4 parts (for a gnomon of 60 parts) at an equinox. This would naturally be rounded to 36 parts, which is a very round number for someone who works in sexagesimal arithmetic. The corresponding latitude is 30° 58′, which is exactly the value that Ptolemy uses. It is a suggestion of this paper that Eratosthenes used this value before him, and for just the same reason.

(4) The arc between Aswan and Alexandria. Eratosthenes assumed that Aswan and Alexandria are on the same meridian, and the error produced by this assumption is small compared with other errors. Thus he took the arc between them to be 30° 58′ minus 23° 51′ (I neglect the seconds here), or 7° 7′. This is 1/50·6 of a circle. Perhaps Eratosthenes simply truncated the denominator to 50; ancient mathematicians often truncated numbers instead of rounding them. Even if he did not do this, it is clear that he was dealing in round numbers in his estimate, and he would choose to round the denominator to 50 rather than 51. In this way, we have derived a possible basis for the angular part of Eratosthenes' measurement of the Earth.

DISCUSSION

Rawlins' work reveals the fact that errors in ancient Greek latitudes are often 1° or more. The errors involved in Eratosthenes' measurement of the Earth's circumference are not as large as this, but they are still too large to have come from using graduated circles. The size of the errors at Alexandria is compatible with the use of a gnomon, but the error in relating Aswan to the tropic is too large for even this method.

However, we reproduce Eratosthenes' data accurately if we assume that they came from old methods of expressing the north–south coordinate of a point. At Alexandria, we assume that the shadow length was rounded from 36·4 parts (for a gnomon of 60 parts) to 36 parts. At Aswan we assume that the length of the longest day was rounded from $13^h\,31^m$ to $13\frac{1}{2}$ hr. We do not actually need Rawlins' hypothesis that the rounding involved in ancient geographical latitudes was done for the convenience of astrologers. The rounding involved in this paragraph would probably have seemed negligible to any scientist of the third century BC.

Thus we have a 'theory' that agrees with all the angular data used in Eratosthenes' measurement, and we should ask whether the agreement is significant or accidential. Let us look first at the obliquity, which he took to be equal to the latitude of Aswan; he made an error of 22 arcmin in doing so.

When a gnomon is used to measure the zenith distance, it is reasonable to take the standard deviation σ to equal 16 arcmin, the radius of the Sun. When a graduated circle is used, the value of σ drops to about 4 arcmin, because the shadow of the Sun is symmetrical in this case. But when a gnomon is used on the tropic at noon on the day of the summer solstice, we also have a symmetrical situation, and the uncertainty in using a gnomon to decide if a point is on the tropic should thus be about 4 arcmin. The error of 22 arcmin is therefore about 5·5 σ, and the probability that an error can be this large is less than 10^{-8}. Thus we have high confidence that the latitude used for Aswan was not measured with a gnomon. Equally, it was not measured with a graduated circle.

Eratosthenes takes the latitude of Aswan to be 23° 51′ 20″. To be conservative, let us ignore the seconds and say that he uses 23° 51′. Our 'theory' predicts that he should have found just this value, namely 23° 51′. The agreement with our theory is exact to the level of rounding used; that is, the agreement is within 30 arcsec*. Now we must ask what are the chances that a person at Aswan would have obtained the value of 23° 51′ by accident. More specifically, we must ask what are the chances that he got a value between 23° 50½′ and 23° 51½′, which he would round to 23° 51′.

The chances are greater if the observer used a gnomon rather than a graduated circle, so I shall consider only the former. If he were actually at Aswan, he was far enough north of the tropic to make the gnomon asymmetrical, so we should take $\sigma = 16$ arcmin. His latitude was 24° 5′, so his error was 14 arcmin. The chance that his error would lie in an interval 1 arcmin wide centred at 14 arcmin is about 0·017, about 1 chance in 60. Thus there is little chance that the agreement happened by chance, and the agreement with our theory is rather significant† even on the conservative basis I have used.

Now let us turn to the latitude β of Alexandria that Eratosthenes used. Here we are hampered by not knowing exactly what β was. All we know is that the ratio $360/(\beta - 23°\ 51')$ is a number that he felt justified in rounding to 50. A moment ago I said that he would probably have felt justified in changing 50·6 to 50, so, for symmetry, let us assume that the ratio in question lay between 49·4 and 50·6. Then β lay in the interval of 10 arcmin between 30° 58′ and 31° 8′, and the average is 31° 3′, an error of 10 arcmin.

If we assume that he used a gnomon, we again have $\sigma = 16$ arcmin. The probability that a measurement would yield a value within the specified range is about 0·20, about 1 part in 5. Thus it is not likely that the value of β came from a careful measurement. It is more likely that it originated in some other way.

Our 'theoretical' value of β is 30° 58′, which corresponds to a shadow length of exactly 36 parts out of 60, and this is within the range that Eratosthenes might have used. The probability that this agreement happened by chance is again about 0·20, about 1 part in 5. Our theory is confirmed by a modest but not overwhelming margin.

However, there is another argument which strengthens the theory, although I do not see how to attach a quantatitive confidence level to the argument. We saw above that Ptolemy claimed to have found 30° 58′ for the latitude by using a graduated meridian circle, and, as we also saw, we may safely conclude that he did not in fact make such a measurement. If he did not find this value from measurement, he must have adopted it either because it was derived from a rounded shadow length, or because it was a traditional value, or both. Since Ptolemy accepted Eratosthenes' value of the obliquity

*The rounding involved in the more precise value of 23° 51′ 20″ was about 5 arcsec, so we are being conservative by a factor of 6.

†I have used the difference between the latitude of Aswan and the value of 23° 51′ in this calculation, rather than the difference between its latitude and the correct position of the tropic. The probability would be still smaller if I used the correct position of the tropic.

without alteration, what is more natural than to assume that he did the same for the latitude?

We should also consider the value of the obliquity briefly. Ptolemy (Chapter I. 12) says that he measured the angle between the tropics several times and that he always found it to be greater than 47° 40′ and less than 47° 45′. This yields 23° 51′ 15″ for the obliquity, which he then alters* to 23° 51′ 20″. This is the value for which the length of the longest day, for a point on the tropic, is exactly 13½ hr. Ptolemy says explicitly that he made these measurements with a graduated circle (or, more probably, a quadrant). I have analysed these claimed observations elsewhere (Newton, 1977, Section V. 6) and shown that it is almost impossible for Ptolemy to have made them. There is little doubt that he simply adopted the obliquity used by Eratosthenes, with no attempt to verify it independently.

Britton (1969) has proposed a method by which Ptolemy might have obtained the value of 23° 51′ 20″, and I have analysed his proposed method in the place just cited. Briefly, Britton suggests that Ptolemy read the zenith distance of the Sun at some time other than noon, and that he calculated the change that would occur between then and noon, with the intention of subtracting it from the measured value. He then, according to Britton, made a mistake and added the correction instead of subtracting it.

It seems unlikely to me that Ptolemy would have adopted such a complicated procedure instead of simply reading the zenith distance at noon, especially after taking the trouble to line his instrument up with the meridian. Even if he did this for some reason, it is highly unlikely that he would have made the mistake in sign every one of the several times he presumably repeated the process. Finally, even if we grant all of these conditions, it is highly unlikely that Ptolemy would have chosen to make his measurements every time at just the time that would make it yield the precise obliquity that Eratosthenes had used, an obliquity that corresponds to a longest day, on the tropic, of exactly 13½ hr.

Since we do not know the method by which Eratosthenes is supposed to have measured the obliquity, the calculations relating to Ptolemy's measurement do not apply exactly. However, as I have shown, the probability that a measurement could lie in some erroneous preassigned interval is small regardless of the details of the measurement process. If the process is highly accurate, it is not likely that it would yield a value that is seriously in error. If the process is inaccurate, it could easily yield an error as large as the one found, but it is unlikely that the error would lie exactly in the narrow preassigned range. Thus it is not likely that the value of 23° 51′ 20″ ever resulted from any careful attempt to measure the obliquity. (See note added in proof.)

In summary, Eratosthenes' measurement of the circumference of the Earth involves two latitudes, those of Aswan and Alexandria. His values of both latitudes are seriously in error and it is not likely that either value came from a careful measurement. On the other hand, both latitudes correspond to values that we calculate precisely from rounded values of north–south coordinates that were used before latitude was introduced as a geographic coordinate. In the case of Aswan, the latitude that Eratosthenes used is the

*Rawlins (1980) suggests this occurred via continued-fraction approximation.

value that corresponds to taking the length of the longest day as exactly 13½ hr. In the case of Alexandria, we do not know the exact latitude that he used. However, if we take the latitude to be the value which makes the equinoctial meridian shadow of a gnomon equal to 36 parts (out of 60, so that $\tan\beta = 0{\cdot}6$), we reproduce the angular separation between Aswan and Alexandria that he used, to satisfactory accuracy. Thus it may be that Eratosthenes' data did not come from any measurements that he made. It may be that they came simply from pre-existing traditions. If so, speculations about the accuracy of his estimate, and about the length of the stade that he used, are essentially meaningless.

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In the version of this paper that I first submitted for publication, I did not make any quantitative estimates of the confidence levels of my hypotheses. I thank one of the referees for his suggestion that I do so.

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NOTE ADDED IN PROOF

Ptolemy actually writes that Eratosthenes took the angle between the tropics to be 11/83 of a full circle, and the ratio 11/83 has excited much comment in the literature. Rawlins (1980) has given what I believe to be the correct explanation of its origin. As he has shown, it was common in Greek mathematics to express an angle by writing it in the form $a \times 360°$ and then expressing a as a continued fraction.

If we use 23° 51′ 20″ for the obliquity, ϵ, the angle between the tropics is 47° 42′ 40″, and $a = 0{\cdot}13253\,08642$. When we expand this as a continued fraction, we get the following typesetter's nightmare:

$$a = 1/\{7 + [1/(1 + [1/\{1 + [1/(5 + [1/195\,..])]\}])]\}.$$

If the reader will rewrite this, using horizontal bars instead of slanting ones to denote fractions, which will allow him to omit the parentheses and brackets, he will find that this expression is not so formidable as it appears here. The first two approximations are 1/7 and 1/8, which are not very accurate. The third approximation is 2/15, which leads to $\epsilon = 24°$. The fourth approximation is the famous 11/83, which leads to $\epsilon = 23°\,51'\,20''$ when rounded to the nearest second. In certain types of computation, it was probably easier to use 11/83 than to use 23° 51′ 20″, which may account for the use of continued fractions.